A TRIAL FOR SYSTEMATIZATION OF METHODS FOR GENERATING AND SOLVING VARIATIONS OF DISCRETE OPTIMIZATION PROBLEMS AND THE THREE JUGS PROBLEM AS AN EXAMPLE

Asen Velchev[1] and Philip Petrov[2]

[1]*University of National and World Economy, Bulgaria, Sofia 1700, Studentski grad "Christo Botev"*
[1]asen_v@abv.bg

[2]*Technical University, Bulgaria, Sofia 1000, Clement Ohridsky 8 blvd*
philip@abv.bg

Abstract

One generalized form of this problem is: "3 vessels of volumes M, N and P contain respectively m, n and $M\text{-}(m+n)$ liters liquid, where $M=N+P$, $N>P$, $m\leq M$, $n\leq N$ and $M\text{-}(m+n)\leq P$. Divide the liquid in two equal parts." The solution is frequently done by trial-error. Perelman provided a "billiard" solution to the problem without prescriptions how to guess to solve it like this. There are numerous articles on this topic, but there are no known to us attempts for exhaustive analysis, synthesis and systematization of them. We consider briefly the most interesting ones with didactics perspective. We prescribe ideas for solutions, generalizations, investigations, generating of new problems and proofs, based on representation of the quantities of liquid in the three vessels as ordered triples - projective (or barycentric) coordinates of points in a Euclidean plane. The pourings are visualized through directed segments between the respective points. This representation provides good visualization for finding all possible solutions, the optimal one in concrete steps, etc. Eventual 3D-representation of the points would lead to more difficulties in the imagination of the spatial objects. The opportunity for systematical approach in seeking for solution enables creation of didactical approach for teaching "how to solve it", including creation of Educational Dialogue Computer Programs. We consider another discrete optimization problem (DOP) with notion how to find the solution. It is an open question in didactics how to guess how to solve a DOP and we make some first steps in dealing with this topic.

Introduction

What is a Discrete Optimization Problem (DOP)? The mention of minimization or maximization is not always bound with Calculus. Some practical problems concern objects of discrete nature – a task to be done in minimum steps, with minimum “resources”, “costs” etc. But there are also non-practical funny or folklore DOP, which people create and solve for intellectual pleasure. DOP are interesting for the skilled students and they may strengthen their motivation for mathematics. A variety of non-practical DOP are frequently given in mathematics contests and Olympiads. But the common thing between practical and non-practical DOP is discrete mathematics.

An exemplary DOP [3] (Canadian mathematical Olympiad 1981):
Six musicians gathered at a chamber music festival. At each scheduled concert some of them played, while the rest listened as members of the audience. What is the least number of such concerts which would be scheduled in order to enable each musician to listen to each of the rest five as a member of the audience?

How to solve a DOP?
The methods of linear programming are not always helpful in discrete optimization. The accepted scheme for solving DOP [3] is the following:

1) ***Boundedness***: Suggest and prove an upper or lower bound of the sought quantity;

2) ***Existence***: Find a case, for which an optimum is reached (***NEW: is it obligatory the upper/lower bound to be this optimum?***).

There is described in [5] an attempt to be shown the necessity of 1) and 2). In [6] and here is given an alternative approach to show the same fact. But here is also shown that the first step is not always obligatory. To solve a DOP one may have to perform a long and purposeful sequence of experiments, analyzing the problem and its details wider and deeper and improving his unsuccessful attempts to solve it. This approach is common to some extent for the majority mathematics problems [1]. According to [3] DOP usually require a variety of knowledge, techniques and skills to find the optimal solution in branches as graph theory, number theory, combinatory, etc. Here we will also discuss the importance of making of ***an appropriate visualization***. The latter may significantly reduce the complexity of the solution and minimize the involved in it theory and objects.

Didactization and importance of DOP: As discrete mathematics is widely used in foundation of computer science, etc, it is advisedly more DOP to be solved by more students. The absence of didactic materials on DOP-solving shows need for "didactization" of it. Some disadvantages of the current DOP-teaching are that:

1) There are practically no DOP without solution among the ones, given to students or the optimal solution always equal to the upper/lower bound. Such DOP is suggested in [6] and variations of the three jugs problem with no solution - here.

2) The prescribed in literature solutions to DOP usually include proofs by examples, not by regularity or a law. There are no questions about the number of the solutions and how to obtain each of them. No requirements to exhaust all possible cases **systematically**. Here are solved DOP exhaustively.

DOP-didactization will also lead to:

1) economy of time and efforts of competitors during their preparation for mathematics competitions,

2) development of systematical approach in DOP solving and

3) making DOP accessible for more students.

All this is very important for students' best performance and width and depth of their knowledge. The systematical finding of all solutions will lead to repetition, generalization, clarification and memorization for long time the strategies for obtaining optimality and developing combinatorial thinking. Long-term DOP solving will help students to develop skills to create optimal schemes, to get optimal decisions, etc. DOP solving usually shows that harmony, uniformity and equal treatment lead to optimality.

In 2005 Google responded with about 24 000 pages related with discrete optimization [4]. In 2009 this number is about 326 000. The increment for these four years is considerable - about 14 times!

It is good to leave the students first to try to solve it by themselves. Reasoning over the task, they will become more and more introduced with it, with the objects in it, with the relations between the statements in the text of it, etc. If someone directly shows to students the shortest way, they will know it without motivation [2]. Attempts for motivation without space for orientation in the task may fall. The suggested solution of the problem for the musicians here is guiding them through sequence of better and *better hypotheses to the goal. Optimal solution students can frequently find when the task is already solved* ***somehow****.*

Solution (new) of the problem for the musicians:

Boundedness: Denote the musicians *A, B, C, D, E* and *F*. Denote *M* the event: "A musician has listened from the audience to a performing colleague once". The desired event is ***G***: "Every musician has listened to each of his colleagues from the audience". Therefore ***G*** will come true when *M* has already come true at least 6.5=30 times as each of the 6 musicians should listen to the rest 5. Let *k* be the number of musicians, playing on the stage during a concert. The rest $6-k$ are listening to them. Thus *M* comes true $k(6-k)$ times. $f(k)=k(6-k)$ reaches maximum 9 for $k=3$. In *n* concerts *M* comes true $9n$ times. As there ***might*** be repeated listenings, then $9n \geq 30$ in order to have ***G***. From $9n \geq 30$ we have $n \geq \frac{10}{3}$, i.e. $n \geq 4$ as *n* is integer. Are four concerts ***sufficient*** to have ***G***? We have 9.4=36 realizations of *M*, but the number of the ***repeated*** listenings between all realizations of *M* ***is unclear***. Is it possible the musicians to be scheduled in four concerts in a way to obtain ***G***? If not, then is it possible for five concerts? Is this method economic and promising to be quick enough? Yes, because for 6 concerts there is a schedule: *ABCDE* to play for *F* in the first concert, then *ABCDF* – for *E*, then *ABCEF* for *D* … Hence we need of ***at most*** two major steps - checking if the optimal number is $n_{opt}=4$ or $n_{opt}=5$. Even if not $n_{opt}=5$, then $n_{opt}=6$ as we have already shown this. Of course, if not $n_{opt}=4$ for $k=3$, then we have to study if $n_{opt}=4$ for $k=2$ (or $k=4$, which is equivalent), because $k(6-k)=8$ and 8.4=32 in this case.

Existence (one visualization): Denote with the ordered pair (*A,B*) the event "*A* has listened to *B*". Hence $(A,B) \neq (B,A)$. Let *ABC* be the performers at the first concert. The result will be:

$$(D,A),(D,B),(D,C)$$
$$(E,A),(E,B),(E,C)$$
$$(F,A),(F,B),(F,C).$$

To avoid repeated pairs, let *ABC* be listeners for the next concert, i.e.:

$$\left(A,D\right),\left(B,D\right),\left(C,D\right)$$
$$\left(A,E\right),\left(B,E\right),\left(C,E\right)$$
$$\left(A,F\right),\left(B,F\right),\left(C,F\right).$$

Thus we have now 18 various ordered pairs. At the third concert there should play two musicians from the one triple, and one - from the other. Let *ABE* be the performers (without loss of generality). It is an element of a **systematical approach** to obtain all isomorphic solutions. Now 4 pairs are new. Definitely we cannot construct a successful scheme this way.
How to improve the first scheme if it is possible? Are there shortcomings in it? It was optimal for the first two concerts, but its effectiveness rapidly decreased further. Let's try with more ***uniformity*** instead of inpatient optimization. Cyclic scheme? If the performers during the four concerts are: *ABC, BCD, CDE* and *DEF*, then we have:

$$\begin{array}{ll} & \left(D,A\right),\left(D,B\right),\left(D,C\right) \\ 1: & \left(E,A\right),\left(E,B\right),\left(E,C\right) \\ & \left(F,A\right),\left(F,B\right),\left(F,C\right), \end{array} \qquad \begin{array}{ll} & \left(A,D\right),\left(A,B\right),\left(A,C\right) \\ 2: & \left(E,D\right),\left(E,B\right),\left(E,C\right) \\ & \left(F,D\right),\left(F,B\right),\left(F,C\right), \end{array}$$

$$\begin{array}{ll} & \left(A,E\right),\left(A,D\right),\left(A,C\right) \\ 3: & \left(B,E\right),\left(B,D\right),\left(B,C\right) \\ & \left(F,E\right),\left(F,D\right),\left(F,C\right), \end{array} \qquad \begin{array}{ll} & \left(A,D\right),\left(B,D\right),\left(C,D\right) \\ 4: & \left(A,E\right),\left(B,E\right),\left(C,E\right) \\ & \left(A,F\right),\left(B,F\right),\left(C,F\right). \end{array}$$

The total number different pairs is $9+5+5+5=24$ now.
What weakness may have this scheme? It didn't treat all the musicians in equal manner. *A* and *E* were playing once, while all the rest participated in three concerts. ***Is this external characteristic essential?*** Yes: musicians who play repeatedly are listened repeatedly by the others. We've ***exhausted*** cyclic schemes of this type. All concerts' permutations give isomorphic solutions. We need to think exhaustively. Only if all possible trials are vain, then we may conclude that $n_{opt} \neq 4$. Is there a scheme in which in every two (subsequent) concerts there is only one repeated musician? Let's try this: *ABC, CDE, EFA* and … *ABC*. Cyclic schemes do not work. ***Is there a scheme treating the all musicians equally? Is it a solution to the problem?*** The participants for four concerts are 3.4=12. The musicians are 6. Hence every musician should play twice. For example: *ABC, ADE, BDF, CEF*. The total number of different ordered

pairs is $9+8+7+6=30$.

Why we need two steps? *In the* ***boundedness*** *step we have proved that* $n \geq 4$. *Hence the optimal value*

$$(*) \qquad n_{opt} \geq 4.$$

In the ***existence*** *step we have constructed a solution for* ***some*** $n=4$. *Hence*

$$(**) \qquad n_{opt} \leq 4.$$

From (*) and (**) it follows that $n_{opt} = 4$. *Here the necessity of both steps is visible. These reasonings are principally the same in all DOP.*

We met no DOP, where the solution is reached for a bit bigger/lesser number than the exact lower/upper limit found in the "boundedness" part, which we consider as a disadvantage of current didactics.

Existence (other visualization):

This solution is based on other visualization - by a 6×6 matrix in the figures 1, 2 and 3 (*A*, *B*, *C* ... are the musicians). The visualization of "*A* has listened to *D*" is by filling the cell $a_{A,D}$:

	A	B	C	D	E	F
A				X		
B						
C						
D						
E						
F						

Fig. 1: *A* has listened to *D*.

With this visualization we have with one glance the full and the empty cells. This enables us to seek for optimal schemes for concerts with the purpose to fill the empty cells and to predict how many concerts are necessary in order to have this. Let the playing musicians in the first two concerts be *ABC*, *DEF*. This scheme as we saw in the previous solution is not optimal. The resulting matrix is:

	A	B	C	D	E	F
A		?		X	X	X
B	?			X	X	X
C				X	X	X
D	X	X	X			
E	X	X	X			
F	X	X	X			

Fig. 2: *ABC* and *DEF* have performed (two concerts).

The cells with question marks may be filled as *A* performs for *B* and then *B* for *A* – hence at least two concerts to fill the matrix. ***Generalization:*** if we have a pair empty orthogonally symmetric to each other cells with respect to the main diagonal, then we need at least two concerts to fill all empty cells. Let us form a schedule of performers-listeners for the third concert. Let *A* performs for *B*. If *A* performs for *C* too, then *BC* are in the audience and they should perform for each other in fourth and ***fifth*** concerts. If *C* is a performer too, the others except *B* do not need to listen to him as his column shows, nor to *A*. If they play, *B* has already listened to them, as his row shows. Let *ADE* perform (and *DEF* in the next concert). I.e.:

	A	B	C	D	E	F
A				x	x	x
B	x		?	x	x	x
C	x	?		x	x	x
D	x	x	x			
E	x	x	x			
F	x	x	x	x	x	

Fig. 3: *ABC, DEF, ADE* have performed (3 concerts).

Question-marks show that at least two concerts are necessary, i.e. at least five totally. It is an isomorphic visualization if we present the musicians as vertices of a hexagon and every realisation of M with a directed segment from the listening to the performing musician or the opposite. The hexagon must be gradually built to a full graph. We prefer matrix visualisation, because if the directed segment $\overrightarrow{AB}$ is on the graph, the opposite $\overrightarrow{BA}$ might be forgotten as the edge *AB* is already present. Another disadvantage of graph representation is the multitude of segments crossing each other (all sides and diagonals of the hexagon). To get ***G*** via matrix one should fill all white cells in it, which is quite easy.

The "three jugs problem" – Discrete Optimization version: "3 jugs of volumes *M*, *N* and *P* contain respectively *m*, *n* and *M-(m+n)* liters liquid, where *M=N+P*, *N >P*. Find the ***minimum number pourings*** necessary to divide the liquid in two equal parts."

The solution is frequently done by trial-error. Polya provides in [1] a purposeful reverse sequence of steps to solve analyze it (and then to solve it). Yakov Perelman provided a "billiard" solution to the problem by construction of a parallelogram of sides *N* and *P* on integer affine coordinate Cartesian net, which internal and contour points represent the quantities of liquid in the second and the third jugs (the rest liquid is in the biggest jug). A better visualization of the quantities of liquid in the three vessels is done by using barycentric coordinates. Here we suggest integer projective coordinates in Euclidean plane for *M=8, N=5, P=3*:

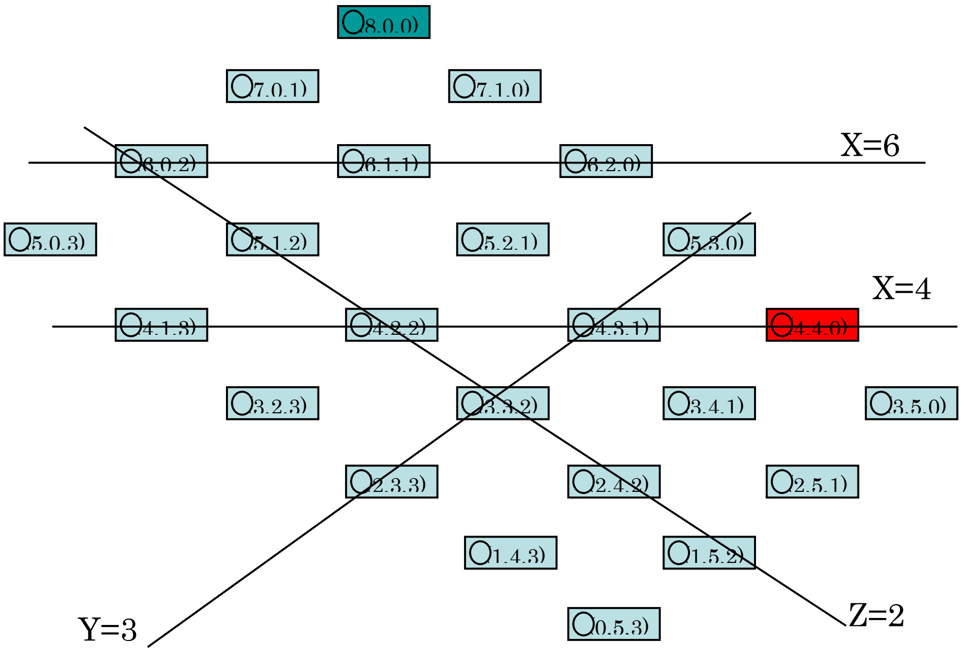

Fig. 4: Visualization of three jugs DOP by integer projective coordinates.

There are shown four lines in ***Fig. 4*** with equations X=4, X=6, Y=3 and Z=2. Each point denotes a state. The first coordinate denotes the quantity of liquid in the biggest jug, the second coordinate – in the second jug and the third – in the smallest one. The liquid is 8 litres constantly, because we do not pour it away but always from one jug to another. The sum of the coordinates of each point is 8. A pouring is a motion along one of these three coordinate directions, because moving along such line leaves exactly one coordinate constant and the others change. This change presents pouring from one to another jug, while the liquid in the third one remains constant. These jugs have no volume thick-marks and therefore we cannot interrupt a pouring until the jug, in which we pour is full. Therefore each motion ends over the contour of the parallelogram bound by the lines Z=0, Y=5, Z=3 and Y=0. Outside the parallelogram some coordinates become negative or the second and third component becomes greater than the respective jug's volume. This visual representation enables easy visualization of Polya's suggestion: the red point (4,4,0) is the final state. It can be reached from (4,1,3), which – ***only*** from (7,1,0), it - ***only*** from (7,0,1), it – etc. Other possibility: (4,4,0) from (1,4,3), … There are no other possibilities. There are 8 steps in the first case and 7 in the second one, which is the optimum. ***Here the "boundedness" step is***

unnecessary! Is it advisable to start with the initial state (8,0,0) and exhaust all possible paths to (4,4,0)? No, because if we do this, the paths are too many as one may explore them for himself.

If we ***modify*** the problem this way: "obtain 4 liters in some jug in minimum steps" instead of "divide the liquid in equal parts", then goal points are seven (on the lines X=4 and Y=4). Internal points drop away and only (4,4,0), (1,4,3) and (4,1,3) can be reached. The first one can be reached in 7 steps; (1,4,3) – in 6 steps, because this point is but last one in the circuit for reaching (4,4,0) in 7 steps. (4,1,3) we can reach in 7 steps analogously. Hence the solution is 6.

If we ***modify*** the generalized version of the problem for M, N, P this way: "divide the liquid in three equal parts", then we obviously need an ***internal*** point $\left(\frac{M}{3},\frac{M}{3},\frac{M}{3}\right)$, i.e. there is ***no solution***.

Conclusion

We've considered different didactical aspects of DOP-solving here. One of the most important approaches appeared to be the discovery of some high-promising visualization(s) of a certain DOP. Future goal is to investigate for how many DOP (or what classes of DOP) exist (or can be discovered) similar high-promising visualizations and how to discover them.

References:

1. Polya G. "How to Solve It" Princeton University Press, Princeton, New Jersey – 1945.
2. Polya G. "With, or without, motivation?" American Mathematical Monthly, v. 56, 1949, p.684-691.
3. Tabov J.B., Taylor P.J: "Methods of problem solving" Australian Mathematics Trust Publishing- 1996.
4. Donev I, Isufov R, Velchev A "Interactive Computer Programs in Distance Education", pp 52–59, proceedings of the second scientific-practical conference "Новите технологии в образованието и професионалното обучение" [The new technologies in education and professional training], Bulgaria, Plovdiv 2004

5. Sybeva N. Some methods for solving extremal tasks in 6th and 7th grade; in Bulgarian: Някои методи за решаване на екстремални задачи в 6. и 7. клас, Математика и информатика, бр. 3, 1999, стр. 15-22
6. Velchev A. Problems in teaching and learning how to solve mathematical problems of the type “discrete optimization” (for mathematics competitions), Conference of young scientists, Sofia, Bulgaria, 2009

Velchev A., Petrov Ph. - "A trial for systematization of methods for generating and solving variations of discrete optimization problems and the three jugs problem as an example", MASSEE International Congress on Mathematics MICOM, Proceedings Education, pages from 214 to 221, ISBN 978-9989-646-40-9, 2009

Reported on MASSEE International Congress on Mathematics MICOM 2009, Ohrid, Republic of Macedonia, 16 – 20 September, 2009